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STABILITY PROBLEM OF SINGULAR STURM-LIOUVILLE EQUATION

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Abstract. We consider the stability of the inverse spectral problems associated with the Sturm-Liouville equation having singularity type $\frac{l(l+1)}{x^2}$. To obtain stability results, we use the method which was given firstly by Ryabushko for regular Sturm-Liouville operator. These results give a bound for the difference between the spectral functions and an estimation for the norm of the difference between the solutions of associated problems under some conditions.

Keywords: singular equation, stability, spectral function.

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1. Introduction

The Sturm-Liouville problem can be completely reconstructed either from its spectral function or from the scattering data and the reconstruction procedures are quite efficient. In particular, they allowed us to find necessary and sufficient conditions for spectral functions of the boundary value problems under consideration. These conditions show that the symmetric boundary value problem is uniquely reconstructed from its spectral function $\rho(\mu)$ given for all μ. It is well known [12] that the spectrum and the norming constants of the operator L define its spectral function and vice versa. The problem of obtaining the potential function from the spectral function of the operator L is solved in the classical study of Gelfand and Levitan [6]. Later, hundreds of publications have been devoted to this subject [1, 3-5, 8-13, 17-20, 24, 26]. This theory provides an effective method recovering the potential function $q(x)$ from the spectra $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$.

A principal question about stability is as follows: what information about the function $q(x)$ or the boundary value problem in general can be obtained, if the spectral function is known only on a finite interval of values of the spectral parameter? To answer this question, one has to know to what extend can two boundary value problems differ from each other, if it is known that their spectral functions differ slightly for λ varying on a finite interval. Local stability of the inverse spectral problem was studied by the authors [7, 16, 21, 25]. Schrodinger equation is solved by numerical methods in [22, 23]. The stability of the inverse problems was proved in [14, 15, 21]. Marchenko and Maslov in [14] deal with stability problem for regular Sturm-Liouville equation in the case of the spectral functions $\rho_j(\lambda)$ coincide on given interval. Aktosun [2] consider stability estimates for potentials in the case of no eigenvalues and when the reflection coefficient is known in some interval. Ryabushko [21] found a bound for the variation of difference between the spectral functions and difference between the solutions of two regular Sturm-Liouville problems. Our approach is much more diffucult than his method in [21], because we applied this method for the singular Sturm-Liouville problem.

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In this paper, we deal with the stability of inverse problems for singular Sturm-Liouville operators. We consider two such problems with potentials $q_1(x)$ and $q_2(x)$ and discuss proximity of their spectral functions and solutions when the first $N+1$ eigenvalues of two spectral problems coincide. We give two important theorems about the stability. In the first theorem, we will give a bound for the variation of the difference between the spectral functions for singular Sturm-Liouville equations with Dirichlet conditions. In the second theorem, we obtain an estimation for the norm of the difference between the solutions for the associated problems.

2. Preliminaries

Consider the singular Sturm-Liouville problem

$$L_1 y = -y'' + \left(q_1(x) + \frac{l(l+1)}{x^2}\right) y = \lambda y, \ 0 < x < 1 \tag{1}$$

with conditions

$$y(0) = 0, \ y(1) = 0 \tag{2}$$

where $q_1(x)$ is a real valued function in $L^2(0,1)$ and l is a nonnegative integer. The operator L_1 is self adjoint on $L^2(0,1)$ and has a discrete spectrum $\{\lambda_{1,n}\}$. The eigenvalues of the problem (1)-(2) coincide with the solutions of $\psi(\lambda, 1) = 0$ and may be given as an increasing sequence $\{\lambda_{1,n}\}_{n=1}^{\infty}$ satisfying

$$\lambda_{1,n} = \left(n + \frac{l}{2}\right)^2 \pi^2 + \int_0^1 q_1(x)\, dx - l(l+1) + a_{1,n}, \tag{3}$$

where $(a_{1,n})_{n\in\mathbb{N}}$ is l_2 sequences (see [9]).

Now, consider the second singular Sturm-Liouville equation corresponding to $q \equiv 0$

$$L_0 y = -y'' + \frac{l(l+1)}{x^2} y = \lambda y \tag{4}$$

with the Dirichlet boundary conditions (2). Denote by $\{\lambda_{0,n}\}$ the eigenvalues of the problem (2), (4) and satisfy

$$\lambda_{0,n} = \left(n + \frac{l}{2}\right)^2 \pi^2 - l(l+1) + a_{0,n}, \tag{5}$$

where $(a_{0,n})_{n\in\mathbb{N}}$ is l_2 sequences [20]. To define norming constants, firstly introduce $\psi(\lambda, x)$ the solution of the equation (1) satisfying the initial conditions

$$\psi(\lambda, 1) = 0 \ , \ \psi'(\lambda, 1) = 1. \tag{6}$$

One has that $\psi(\lambda, x)$ is regular on $(0, 1]$ with $\psi(\lambda, x) = O\left(x^{-l}\right)$, as $x \to 0$ unless $\lambda = \lambda_n$ is a Dirichlet eigenvalue. We denote the norming constants of the problem (1)-(2) by

$$\alpha_{1,n} = \int_0^1 \psi^2(\lambda_n, x)\, dx.$$

It's well known that the representation of the norming constants $\alpha_{1,n}$ by two spectra is given

$$\alpha_{1,n} = \frac{(n\pi)^{\frac{l}{2}} (-1)^{-n+1} j_l\left(\sqrt{\lambda_{1,n}}\right)}{\lambda^{\frac{l}{2}} (\lambda_{0,n} - \lambda_{1,n})} \prod_{m\neq n} \frac{\lambda_{1,m} - \lambda_{1,n}}{\lambda_{0,m} - \lambda_{1,n}}$$

for every $n \in \mathbb{N}$,[1] where j_l is the usual spherical Bessel functions in [9].

(In case $\lambda_n = \lambda_{0,k}$ for some $k \in \mathbb{N}$, we must replace the fraction $\frac{j_l(\sqrt{\lambda_{1,n}})}{\lambda_{0,k}-\lambda_n}$ with the limiting expression $-\frac{j_l'(\sqrt{\lambda_{0,k}})}{2\sqrt{\lambda_{0,k}}}$.)[1]

Let us introduce different singular Sturm-Liouville equation

$$L_2 y = -y'' + \left(q_2(x) + \frac{l(l+1)}{x^2}\right) y = \lambda y \, , \, 0 < x < 1 \tag{7}$$

with boundary conditions (2), where $q_2(x)$ is a real valued function and $q_2 \in L^2(0,1)$. The operator L_2 is self adjoint on $L^2(0,1)$ and has a discrete spectrum $\{\lambda_{2,n}\}$ which conform to the classical asymptotics

$$\lambda_{2,n} = \left(n + \frac{l}{2}\right)^2 \pi^2 + \int_0^1 q_2(x)\, dx - l(l+1) + a_{2,n}, \tag{8}$$

where $(a_{2,n})_{n\in\mathbb{N}}$ is l_2 sequences. Let us denote the solution of equation (7) by $\varphi(\lambda, x)$ satisfying the initial conditions (6).

Similarly, denote the norming constants of the problem (2), (7) respect to two spectra for every $n \in \mathbb{N}$, by

$$\alpha_{2,n} = \frac{(n\pi)^{\frac{l}{2}} (-1)^{-n+1} j_l\left(\sqrt{\lambda_{2,n}}\right)}{\lambda^{\frac{l}{2}} (\lambda_{0,n} - \lambda_{2,n})} \prod_{m\neq n} \frac{\lambda_{2,m} - \lambda_{2,n}}{\lambda_{0,m} - \lambda_{2,n}}.$$

Set the spectral functions of the problem (1)–(2) and (2), (7) by

$$\rho_1(\lambda) = \sum_{\lambda_{1,n}<\lambda} \frac{1}{\alpha_{1,n}},$$

$$\rho_2(\lambda) = \sum_{\lambda_{2,n}<\lambda} \frac{1}{\alpha_{2,n}},$$

respectively.

3. Main results

In this section, we give main theorems about the stability of the spectral functions and solutions respect to two spectra. More exactly, we evaluate the variation of the difference between the spectral functions and obtain an estimation of difference between two solutions of problems (1)–(2) and (2), (7) when the eigenvalues $\{\lambda_{j,m}\}$, $(j = 1, 2)$ of these problems coincide numbers of $N+1$. Before we give main theorems, let give following lemma containing asymptotics of the solutions.

Lemma 3.1. *Let $q'(x) \in L_1(0,1)$, then the following inequalities hold*

$$\left| \left(\psi(\lambda, x) + \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}} \right) e^{i\sqrt{\lambda}(1-x)} \right| \le \frac{\sigma(x)}{\left|\sqrt{\lambda}\right| \left(\left|\sqrt{\lambda}\right| - \sigma(x) \right)}, \tag{9}$$

$$\left| \left(\psi(\lambda, x) + \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}} - \frac{\cos\sqrt{\lambda}(1-x)}{\lambda} \int_x^1 \left(q_1(t) + \frac{l(l+1)}{t^2} \right) dt \right) e^{i\sqrt{\lambda}(1-x)} \right|$$

$$< \frac{1}{|\lambda|^{3/2}} \left(\frac{1}{2}\sigma_{-1}(x) + \frac{\sigma^2(x)}{1 - \frac{\sigma(x)}{|\sqrt{\lambda}|}} \right) \tag{10}$$

for $\left|\sqrt{\lambda}\right| > \sigma(x)$ *and* $Im\sqrt{\lambda} \geq 0$, *where*

$$\sigma(x) = \int_x^1 \left| q_1(t) + \frac{l(l+1)}{t^2} \right| dt, \quad \sigma_{-1}(x) = \int_x^1 \left| q_1(t) + \frac{l(l+1)}{t^2} \right|' dt.$$

Proof. It is easy to verify that the function $\psi(\lambda, x)$ satisfies the following integral equation:

$$\psi(\lambda, x) = -\frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}} + \int_x^1 \frac{\sin\sqrt{\lambda}(t-x)}{\sqrt{\lambda}} \psi(\lambda, t) \left(q_1(t) + \frac{l(l+1)}{t^2} \right) dt.$$

To show that the inequality (9) holds, consider the function $\tau(\lambda, x)$ defined as follows:

$$\tau(\lambda, x) = \left\{ \psi(\lambda, x) + \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}} \right\} e^{i\sqrt{\lambda}(1-x)}, \quad \left(Im\sqrt{\lambda} \geq 0 \right)$$

It is obvious that we have

$$\begin{aligned} \tau(\lambda, x) &= \int_x^1 \frac{\sin\sqrt{\lambda}(t-x)}{\sqrt{\lambda}} \tau(\lambda, t) e^{i\sqrt{\lambda}(t-x)} \left(q_1(t) + \frac{l(l+1)}{t^2} \right) dt \\ &\quad -\frac{1}{\lambda} \int_x^1 \sin\sqrt{\lambda}(t-x) \sin\sqrt{\lambda}(1-t) e^{i\sqrt{\lambda}(t-x)} \\ &\qquad \times e^{i\sqrt{\lambda}(1-t)} \left(q_1(t) + \frac{l(l+1)}{t^2} \right) dt. \end{aligned} \tag{11}$$

Denote

$$m(\lambda, x) = \max_{0 \leq x \leq t} |\tau(\lambda, t)|.$$

It is known that [21] we have

$$\left| \frac{\sin\sqrt{\lambda}(t-x)}{\sqrt{\lambda}} e^{i\sqrt{\lambda}(t-x)} \right| < \frac{1}{\left|\sqrt{\lambda}\right|}, \left(t > x \geq 0, Im\sqrt{\lambda} \geq 0 \right),$$

$$\left| \cos\sqrt{\lambda}(1-x) e^{i\sqrt{\lambda}(1-x)} \right| < 1, \left(Im\sqrt{\lambda} \geq 0 \right).$$

Considering the above inequalities, rewrite equation (11) as

$$m(\lambda, x) \leq m(\lambda, x) \frac{\int_x^1 \left| q_1(t) + \frac{l(l+1)}{t^2} \right| dt}{\left|\sqrt{\lambda}\right|} + \frac{\int_x^1 \left| q_1(t) + \frac{l(l+1)}{t^2} \right| dt}{|\lambda|}.$$

Hence, for $\left|\sqrt{\lambda}\right| > \sigma(x)$ the last inequality gives Equation (9). Let us prove Equation (10). If we consider following integral equation

$$-\int_x^1 \frac{\sin\sqrt{\lambda}(t-x)}{\sqrt{\lambda}} \frac{\sin\sqrt{\lambda}(1-t)}{\sqrt{\lambda}} \left(q_1(t) + \frac{l(l+1)}{t^2} \right) dt,$$

it is easy to verify that last integral equation is equal to

$$\frac{\cos\sqrt{\lambda}(1-x)}{2\lambda} \int_x^1 \left(q_1(t) + \frac{l(l+1)}{t^2} \right) dt - \frac{1}{2\lambda} \int_x^1 \cos\sqrt{\lambda}(2t-x-1) \left(q_1(t) + \frac{l(l+1)}{t^2} \right) dt. \tag{12}$$

Here, applying the partial integration method to the second integral equation at the right side of (12), we obtain

$$\begin{aligned}
&\int_x^1 \cos\sqrt{\lambda}\,(2t-x-1)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)dt \\
= &\left(q_1(t)+\frac{l(l+1)}{t^2}\right)\frac{\sin\sqrt{\lambda}\,(2t-x-1)}{2\sqrt{\lambda}}\Big|_x^1 \\
&-\frac{1}{2\sqrt{\lambda}}\int_x^1 \sin\sqrt{\lambda}\,(2t-x-1)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)' dt.
\end{aligned}$$

The last equality imply that

$$\begin{aligned}
&-\int_x^1 \frac{\sin\sqrt{\lambda}\,(t-x)}{\sqrt{\lambda}}\frac{\sin\sqrt{\lambda}\,(1-t)}{\sqrt{\lambda}}\left(q_1(t)+\frac{l(l+1)}{t^2}\right)dt \\
= \ &\frac{\cos\sqrt{\lambda}\,(1-x)}{2\lambda}\int_x^1\left(q_1(t)+\frac{l(l+1)}{t^2}\right)dt \\
&-\frac{\sin\sqrt{\lambda}\,(1-x)}{4\lambda^{3/2}}\left(q_1(1)+l(l+1)+q_1(x)+\frac{l(l+1)}{x^2}\right) \\
&+\frac{1}{4\lambda^{3/2}}\int_x^1 \sin\sqrt{\lambda}\,(2t-x-1)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)' dt \\
< \ &\frac{\cos\sqrt{\lambda}\,(1-x)}{2\lambda}\int_x^1\left(q_1(t)+\frac{l(l+1)}{t^2}\right)dt \\
&+\frac{1}{4\lambda^{3/2}}\int_x^1\left\{\sin\sqrt{\lambda}\,(2t-x-1)+\sin\sqrt{\lambda}\,(1-x)\right\}\left(q_1(t)+\frac{l(l+1)}{t^2}\right)' dt.
\end{aligned}$$

Now, if we put last inequality in Equation (11) we get

$$\begin{aligned}
\tau(\lambda,x) \ < \ &\int_x^1 \frac{\sin\sqrt{\lambda}\,(t-x)}{\sqrt{\lambda}}\tau(\lambda,t)\,e^{i\sqrt{\lambda}(t-x)}\left(q_1(t)+\frac{l(l+1)}{t^2}\right)dt \\
&+\frac{e^{i\sqrt{\lambda}(1-x)}}{4\lambda^{3/2}}\int_1^x\left\{\sin\sqrt{\lambda}\,(1-x)+\sin\sqrt{\lambda}\,(2t-x-1)\right\}\left(q_1(t)+\frac{l(l+1)}{t^2}\right)' dt \\
&+\frac{\cos\sqrt{\lambda}\,(1-x)\,e^{i\sqrt{\lambda}(1-x)}}{2\lambda}\int_x^1\left(q_1(t)+\frac{l(l+1)}{t^2}\right)dt.
\end{aligned}$$

Finally, using the result of Equation (9) we get desired estimate

$$\left|\left\{\psi(\lambda,x)+\frac{\sin\sqrt{\lambda}\,(1-x)}{\sqrt{\lambda}}-\frac{\cos\sqrt{\lambda}\,(1-x)}{2\lambda}\int_x^1\left(q_1(t)+\frac{l(l+1)}{t^2}\right)dt\right\}e^{i\sqrt{\lambda}(1-x)}\right|$$

$$
\begin{aligned}
&< \frac{m(\lambda, x)}{\left|\sqrt{\lambda}\right|} \int\limits_x^1 \left| q_1(t) + \frac{l(l+1)}{t^2} \right| dt + \frac{1}{2|\lambda|^{3/2}} \int\limits_x^1 \left| q_1(t) + \frac{l(l+1)}{t^2} \right|' dt \\
&< \frac{1}{|\lambda|^{3/2}} \left\{ \frac{\sigma^2(x)}{1 - \frac{\sigma(x)}{\left|\sqrt{\lambda}\right|}} + \frac{1}{2}\sigma_{-1}(x) \right\}.
\end{aligned}
$$

Hence the proof of Lemma 3.1 is completed. □

Now, we give main theorem in this study.

Theorem 3.1. *Let the eigenvalues* $\{\lambda_{j,m}\}, (j = 1, 2)$ *of problems (1)–(2) and (2), (7) coincide the numbers of* $N + 1$, *that is,* $\lambda_{1,m} = \lambda_{2,m}$ *for* $m = 1, 2, ..., N + 1$ *and the eigenvalues* $\{\lambda_{0,m}\}$ *corresponding to* $q \equiv 0$ *of these problems are equal to each other then*

$$
\underset{-\infty<\lambda<\frac{N}{2}}{Var} \{\rho_1(\lambda) - \rho_2(\lambda)\} < \rho_1\left(\tfrac{N}{2}\right) \frac{8A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2 N^2} e^{\frac{3A\left(1+\frac{l+3}{2N}\right)^2}{N^2\pi^2}}
$$

for $m > N + 1$, $n < \frac{N}{2}$ *and* $N \geq 2\sqrt{A}$ *where*

$$
A = \int\limits_0^1 |q_2(t) - q_1(t)|\, dt + O\left(\frac{1}{m^2}\right).
$$

Proof. Consider the difference of the spectral functions

$$
\rho_1(\lambda) - \rho_2(\lambda) = \sum_{\lambda_n<\lambda} \frac{1}{\alpha_{1,n}} \left(1 - \frac{\alpha_{1,n}}{\alpha_{2,n}}\right),
$$

where

$$
1 - \frac{\alpha_{1,n}}{\alpha_{2,n}} = 1 - \frac{j_l\left(\sqrt{\lambda_{1,n}}\right)(\lambda_{0,n} - \lambda_{2,n})}{j_l\left(\sqrt{\lambda_{2,n}}\right)(\lambda_{0,n} - \lambda_{1,n})} \prod_{m\neq n}^{\infty} \frac{(\lambda_{1,m} - \lambda_{1,n})(\lambda_{0,m} - \lambda_{2,n})}{(\lambda_{2,m} - \lambda_{2,n})(\lambda_{0,m} - \lambda_{1,n})}.
$$

By definition of the variation, we have for $\lambda_0 < \lambda_{N+2}$,

$$
\underset{-\infty<\lambda<\lambda_0}{Var} \{\rho_1(\lambda) - \rho_2(\lambda)\} \leq \max_{\lambda_n<\lambda_0} \left|1 - \frac{\alpha_{1,n}}{\alpha_{2,n}}\right| \sum_{\lambda_n<\lambda_0} \frac{1}{\alpha_{1,n}} = \rho_1(\lambda_0) \max_{\lambda_n<\lambda_0} \left|1 - \frac{\alpha_{1,n}}{\alpha_{2,n}}\right|. \tag{13}
$$

Hence, to evaluate the difference of the spectral functions for $\lambda_0 < \lambda_{N+2}$, let consider the absolute value at the right side of the Equation (13) as follows:

$$
\max_{n<\frac{N}{2}} \left|1 - \frac{\alpha_{1,n}}{\alpha_{2,n}}\right| = \max_{n<\frac{N}{2}} \left|1 - \prod_{m=N+2}^{\infty} \frac{(\lambda_{1,m} - \lambda_{1,n})(\lambda_{0,m} - \lambda_{2,n})}{(\lambda_{2,m} - \lambda_{2,n})(\lambda_{0,m} - \lambda_{1,n})}\right|. \tag{14}
$$

Considering the infinite product

$$
\Psi(\lambda_n) = \prod_{m=N+2}^{\infty} \frac{(\lambda_{1,m} - \lambda_{1,n})(\lambda_{0,m} - \lambda_{2,n})}{(\lambda_{2,m} - \lambda_{2,n})(\lambda_{0,m} - \lambda_{1,n})},
$$

it follows that

$$
\begin{aligned}
|\ln \Psi(\lambda_n)| &= \left| \sum_{m=N+2}^{\infty} \ln\left(\frac{\lambda_{1,m} - \lambda_{1,n}}{\lambda_{2,m} - \lambda_{2,n}}\right) + \sum_{m=N+2}^{\infty} \ln\left(\frac{\lambda_{0,m} - \lambda_{2,n}}{\lambda_{0,m} - \lambda_{1,n}}\right) \right| \\
&\leq \sum_{m=N+2}^{\infty} \left| \ln\left(1 - \frac{\lambda_{2,m} - \lambda_{1,m}}{\lambda_{2,m} - \lambda_{1,n}}\right) \right| + \sum_{m=N+2}^{\infty} \left| \ln\left(1 - \frac{\lambda_{2,n} - \lambda_{1,n}}{\lambda_{0,m} - \lambda_{1,n}}\right) \right|. \qquad (15)
\end{aligned}
$$

It can be easily seen

$$\left|\frac{\lambda_{2,m}-\lambda_{1,m}}{\lambda_{2,m}-\lambda_{1,n}}\right|<1 \text{ and } \left|\frac{\lambda_{2,n}-\lambda_{1,n}}{\lambda_{0,m}-\lambda_{1,n}}\right|<1$$

for $m>N+1$ and $n<\frac{N}{2}$. It is obvious that the following inequality holds

$$\ln(1-z)<\frac{|z|}{1-|z|}$$

for $|z|<1$. The last inequality implies that

$$|\ln\Psi(\lambda_n)|<\sum_{m=N+2}^{\infty}\frac{\left|\frac{\lambda_{2,m}-\lambda_{1,m}}{\lambda_{2,m}-\lambda_{1,n}}\right|}{1-\left|\frac{\lambda_{2,m}-\lambda_{1,m}}{\lambda_{2,m}-\lambda_{1,n}}\right|}+\sum_{m=N+2}^{\infty}\frac{\left|\frac{\lambda_{2,n}-\lambda_{1,n}}{\lambda_{0,m}-\lambda_{1,n}}\right|}{1-\left|\frac{\lambda_{2,n}-\lambda_{1,n}}{\lambda_{0,m}-\lambda_{1,n}}\right|}. \tag{16}$$

Here, using the asymptotic formulas of the eigenvalues (3), (5) and (8), we have

$$\begin{aligned}\left|\frac{\lambda_{2,m}-\lambda_{1,m}}{\lambda_{2,m}-\lambda_{1,n}}\right| &< \frac{\left|\int_0^1\{q_2(t)-q_1(t)\}\,dt+a_{2,m}-a_{1,m}\right|}{\left|\lambda_{2,m}\left(1-\frac{\lambda_{1,n}}{\lambda_{2,m}}\right)\right|}\\ &< \frac{\left|\int_0^1\{q_2(t)-q_1(t)\}\,dt+a_{2,m}-a_{1,m}\right|}{\left|\lambda_{2,m}\left(1-\frac{\lambda_{1,[\![\frac{N}{2}]\!]}}{\lambda_{2,N+2}}\right)\right|}\\ &< \frac{4A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2\left(1+\frac{5l+13}{3N}+\frac{(5l+7)(l+3)}{3N^2}\right)}\\ &< \frac{4A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2}\end{aligned} \tag{17}$$

and

$$\left|\frac{\lambda_{2,n}-\lambda_{1,n}}{\lambda_{0,m}-\lambda_{1,n}}\right|<\left|\frac{\int_0^1\{q_2(t)-q_1(t)\}\,dt+a_{2,n}-a_{1,n}}{\lambda_{0,m}\left(1-\frac{\lambda_{1,n}}{\lambda_{0,m}}\right)}\right|<\frac{4A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2}$$

for $m>N+1$ and $n<\frac{N}{2}$. Further, substituting (17) and (18) into (16), we have for $N\geq 2\sqrt{A}$

$$\begin{aligned}|\ln\Psi(\lambda_n)| &< 2\frac{\frac{4A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2}}{1-\frac{4A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2}}<\frac{\frac{4A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2}}{1-\frac{4A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^24A}}\\ &< \frac{8A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2}.\end{aligned}$$

By virtue of (14), it follows that

$$\max_{n<\frac{N}{2}}\left|1-\frac{\alpha_{1,n}}{\alpha_{2,n}}\right|<e^{\frac{8A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2}}-1.$$

Thanks to the serial expansion of the exponential function, the last inequality yields

$$\max_{n<\frac{N}{2}}\left|1-\frac{\alpha_{1,n}}{\alpha_{2,n}}\right|<\frac{8A\left(1+\frac{l+3}{2N}\right)^2}{3\pi^2N^2}e^{\frac{3A\left(1+\frac{l+3}{2N}\right)^2}{N^2\pi^2}}. \tag{18}$$

By considering (13) and (18), we obtain the proof of the Theorem 3.1. □

Now, we derive convenient representations for the difference of the solutions of problems (1)–(2) and (2), (7). Let $\psi(\lambda, x)$ and $\varphi(\lambda, x)$ denote the solutions of equations (1) and (7) satisfying the initial conditions (6), respectively.

Theorem 3.2. *The following formula holds*

$$|\psi(\lambda, x) - \varphi(\lambda, x)|^2 < \frac{4x}{N} \exp\left\{\sigma_1(x) - \sigma_1\left(x + \frac{1}{\sqrt{\lambda}}\right) + \sigma_2(x) - \sigma_2\left(x + \frac{1}{\sqrt{\lambda}}\right)\right\}$$
$$\times \exp\{\beta_1(x) + \beta_2(x)\}$$
$$\times \rho_1\left(\frac{N}{2}\right) \frac{8A\left(1 + \frac{l+3}{2N}\right)^2}{3\pi^2 N^2} e^{\frac{3A\left(1+\frac{l+3}{2N}\right)^2}{N^2\pi^2}} \tag{19}$$

under the conditions of Theorem 3.1 and for $0 \le \lambda \le \frac{N}{2}$, *where*

$$\beta_i(x) = \int_1^x \left|q_i(t) + \frac{l(l+1)}{t^2}\right| dt + \left|Im\sqrt{\lambda}\right|(1-x), \ (i = 1, 2)$$

$$\sigma_i(x) = \int_x^1 \int_\tau^1 \left|\left(q_i(t) + \frac{l(l+1)}{t^2}\right)\right| dt d\tau, \ (i = 1, 2).$$

Proof. We have the following equality ([14], given 3.21)

$$[\psi(\lambda, x) - \varphi(\lambda, x)]^2 = \varphi(\lambda, x) \int_0^{\frac{N}{2}} \varphi(\mu, x)\, d\rho_{1,2}(\mu) \int_0^x \psi(\mu, t)\, \psi(\lambda, t)\, dt$$
$$-\psi(\lambda, x) \int_0^{\frac{N}{2}} \psi(\mu, x)\, d\rho_{1,2}(\mu) \int_0^x \varphi(\mu, t)\, \varphi(\lambda, t)\, dt$$
$$+ \int_0^x q_{1,2}(t)\, dt$$
$$\times \int_{\frac{N}{2}}^{\infty} \frac{\{\psi(\lambda, t)\, \varphi(\lambda, t)\psi(\mu, x)\, \varphi(\mu, x) - \psi(\lambda, x)\, \varphi(\lambda, x)\psi(\mu, t)\, \varphi(\mu, t)\}}{\mu - \lambda} d\rho_{1,2}(\mu). \tag{20}$$

Here the interval is taken $I = \left(0, \frac{N}{2}\right)$ instead of $I = (a, b)$.

The solutions $\psi(\lambda, x)$ and $\varphi(\lambda, x)$ can be approximated by the form

$$\psi(\lambda, x) = \frac{\sin\sqrt{\lambda}(x-1)}{\sqrt{\lambda}} + \int_x^1 \frac{\sin\sqrt{\lambda}(t-x)}{\sqrt{\lambda}} \psi(\lambda, t)\left(q_1(t) + \frac{l(l+1)}{t^2}\right) dt,$$

$$\varphi(\lambda, x) = \frac{\sin\sqrt{\lambda}(x-1)}{\sqrt{\lambda}} + \int_x^1 \frac{\sin\sqrt{\lambda}(t-x)}{\sqrt{\lambda}} \varphi(\lambda, t)\left(q_2(t) + \frac{l(l+1)}{t^2}\right) dt,$$

respectively, [24]. These solutions satisfy following inequalities

$$|\psi(\lambda, x)| \le k(\lambda, x) \exp\left\{\sigma_1(x) - \sigma_1\left(x + \frac{1}{\sqrt{\lambda}}\right)\right\}, \ \lambda > 0 \tag{21}$$

$$|\varphi(\lambda,x)| \leq k(\lambda,x)\exp\left\{\sigma_2(x)-\sigma_2\left(x+\frac{1}{\sqrt{\lambda}}\right)\right\},\ \lambda>0 \tag{22}$$

where $k(\lambda,x)=\min\left(x,\frac{1}{\sqrt{\lambda}}\right)$.

The inequalities (21) and (22) can be obtained by Picard's iteration method:

$$\psi(\lambda,x)=\sum_{n=0}^{\infty}\psi_n(\lambda,x),$$

where

$$\psi_0(\lambda,x)=\frac{\sin\sqrt{\lambda}(x-1)}{\sqrt{\lambda}}$$

and

$$\psi_{n+1}(\lambda,x)=\int_x^1\frac{\sin\sqrt{\lambda}(t-x)}{\sqrt{\lambda}}\psi_n(\lambda,t)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)dt.$$

We obtain

$$|\psi_0(\lambda,x)|\leq k(\lambda,x),$$

$$\begin{aligned}|\psi_{n+1}(\lambda,x)| &\leq \int_x^{x+\frac{1}{\sqrt{\lambda}}}\left|(t-x)\psi_n(\lambda,t)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)\right|dt\\ &+\frac{1}{\sqrt{\lambda}}\int_{x+\frac{1}{\sqrt{\lambda}}}^1\left|\psi_n(\lambda,t)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)\right|dt.\end{aligned}$$

Let introduce

$$\begin{aligned}\phi_n(\lambda,x)= &\int_x^{x+\frac{1}{\sqrt{\lambda}}}\left|(t-x)\psi_n(\lambda,t)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)\right|dt\\ &+\frac{1}{\sqrt{\lambda}}\int_{x+\frac{1}{\sqrt{\lambda}}}^1\left|\psi_n(\lambda,t)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)\right|dt.\end{aligned} \tag{23}$$

From (23), we get

$$\phi_n'(\lambda,x)=\int_x^{x+\frac{1}{\sqrt{\lambda}}}\left|\psi_n(\lambda,t)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)\right|dt.$$

The comparison of the last two equalities yields

$$|\psi_{n+1}(\lambda,x)|\leq\phi_n(\lambda,x)=\int_x^1\int_\tau^{\tau+\frac{1}{\sqrt{\lambda}}}\left|\psi_n(\lambda,t)\left(q_1(t)+\frac{l(l+1)}{t^2}\right)\right|dtd\tau. \tag{24}$$

The function $\phi_n(\lambda,x)$ satisfies the initial condition

$$\phi_n(\lambda,1)=0.$$

By induction in (24), it is easily seen that

$$|\psi_n(\lambda, x)| \leq k(\lambda, x) \frac{\left\{\sigma_1(x) - \sigma_1\left(x + \frac{1}{\sqrt{\lambda}}\right)\right\}^n}{n!}.$$

From the last inequality, we arrive at the formula (21). By similar way, Equation (22) can be obtained.

Furthermore, if we consider the equations (1) and (7) with initial conditions (6), then the solutions of these problems satisfy

$$\left|\psi(x, \lambda) + \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}}\right| \leq \frac{\exp\left\{\int\limits_1^x \left(|q_1(t)| + \frac{|l(l+1)|}{t^2}\right)dt + \left|Im\sqrt{\lambda}\right|(1-x)\right\}}{\sqrt{\lambda}}, \tag{25}$$

$$\left|\varphi(x, \lambda) + \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}}\right| \leq \frac{\exp\left\{\int\limits_1^x \left(|q_2(t)| + \frac{|l(l+1)|}{t^2}\right)dt + \left|Im\sqrt{\lambda}\right|(1-x)\right\}}{\sqrt{\lambda}} \tag{26}$$

for $\left|\sqrt{\lambda}\right| \geq 1$, respectively, [5].

Next, using the formula (20) and taking the interval $I = \left(0, \frac{N}{2}\right)$, we obtain

$$|\psi(\lambda, x) - \varphi(\lambda, x)|^2 \leq \left|\varphi(\lambda, x) \int\limits_0^{\frac{N}{2}} \varphi(\mu, x)\, d\rho_{1,2}(\mu) \int\limits_0^x \psi(\mu, t)\,\psi(\lambda, t)\, dt \right.$$
$$\left. -\psi(\lambda, x) \int\limits_0^{\frac{N}{2}} \psi(\mu, x)\, d\rho_{1,2}(\mu) \int\limits_0^x \varphi(\mu, t)\,\varphi(\lambda, t)\, dt \right| \tag{27}$$

for $0 \leq \lambda \leq \frac{N}{2}$ [21]. If we put (21), (22), (25) and (26) into (27), we get

$$|\psi(\lambda, x) - \varphi(\lambda, x)|^2 \leq \frac{2k^2(\lambda, x)}{\lambda} \exp\left\{\sigma_1(x) - \sigma_1\left(x + \frac{1}{\sqrt{\lambda}}\right) + \sigma_2(x) - \sigma_2\left(x + \frac{1}{\sqrt{\lambda}}\right)\right\}$$
$$\times \exp\left\{\beta_1(x) + \beta_2(x)\right\} \int\limits_0^{\frac{N}{2}} d\rho_{1,2}(\mu).$$

Finally, using the result of Theorem 3.1, we obtain Equation (19).

Therefore the proof is completed. □

4. Conclusions

In conclusion, we have emphasized the importance of a certain stability of the inverse singular Sturm-Liouville problems. By Ryabushko's method, we have showed the proximity of the spectral functions and the solutions of two spectral problems (1), (2) and (2), (7) when their eigenvalues coincide finitely.

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References

[1] Albeverio, S., Hryniv, R., Mykytyuk, Ya., (2007), Inverse spectral problems for coupled oscillating systems: Reconstruction from three spectra, Methods Funct. Anal. Topology, 13(2), pp.110-123.

[2] Aktosun, T., (1987), Stability of the Marchenko inversion, Inverse Problems, 3(4), pp.555-563.

[3] Bas, E., Panakhov, E. S., Yilmazer, R., (2013), The Uniqueness theorem for hydrogen Atom equation, TWMS J. Pure Appl. Math., 4(1), pp.20-28.

[4] Carlson, R., (1993), Inverse spectral theory for some singular Sturm-Liouville problems, J. Diff. Eq., 106, pp.121-140.

[5] Carlson, R., (1997), A Borg-Levinson theorem for Bessel operator, Pasific J. Math., 177, pp.1-26.

[6] Gelfand, I. M., Levitan, B. M., (1951), On the determination of a differential equations by its spectral function, Izv. Akad. Nauk. SSSR, Ser. Math., 15, pp.309-360.

[7] Gulsen, T., Yilmaz, E., Panakhov, E. S., (2017), On a Lipschitz stability problem for p-Laplasian Bessel equation, Commun. Fac. Sci. Univ. Ank. Series A1, 66(2), pp.253-262.

[8] Hochstadt, H., (1973), The Inverse Sturm-Liouville problem, Comm. Pure Appl. Math., 26, pp.715-729.

[9] Hryniv, R., Sacks, P., (2010), Numerical solution of the inverse spectral problem for Bessel operators, J. Comput. Appl. Math., 235(1), pp.120-136.

[10] Levinson, N., (1949), The inverse Sturm- Liouville problem, Math. Tidsskr. B., 25, pp.25-30.

[11] Levitan, B.M., Gasymov, M.G., (1964), The determination of a differential equation from two spectra, Uspekhi Mat. Nauk., 19, pp.3-63.

[12] Levitan, B.M., (1978), On the Determination of the Sturm-Liouville operator from one and two spectra, Math. Ussr., Izvestija, 12(1), pp.179-193.

[13] Marchenko, V. A., (1952), Certain problems of the theory of one dimensional linear differential operators of the second order, Trudy Moskovskogo Matematicheskogo Obshchestva, 1, pp. 327-340.

[14] Marchenko, V.A., Maslov, K.V., (1970), Stability of the problem of reconstruction of the Sturm-Liouville operator in terms of the spectral function, Mathematics of the USSR Sbornik, 81, pp.525-51.(in Russian)

[15] Marletta, M., Weikard, R., (2007), Stability for the inverse resonance problem for a Jacobi operator with complex potential, Inverse Problems, 23, pp. 1677-88.

[16] McLaughlin, J.R., (1988), Stability theorems for two inverse problems, Inverse Problems, IOPscience pp.4529-40.

[17] Panakhov, E.S., Sat, M., (2013), Inverse problem for the interior spectral data of the equation of hydrogen atom, Ukrainian Math. J., 64(11), pp.1716-1726.

[18] Panakhov, E.S., Koyunbakan, H., (2003), Inverse problem for singular Sturm-Liouville operator, Proceeding of IMM of NAS of Azerbaijan, 18(26), pp.113-126.

[19] Panakhov, E.S., Bas E., (2012), Inverse problem having special singularity type from two spectra, Tamsui Oxf. J. Inf. Math. Sci, 28(3), pp.239-258.

[20] Rundell, W., Sacks, P.E., (2001), Reconstruction of a radially symmetric potential from two spectral sequences, J. Math. Anal. Appl., 264(2), pp.354-381.

[21] Ryabushko, T.I., (1973), Stability of the reconstruction of a Sturm-Liouville operator from two spectra, II Teor. Funkts. Anal. Prilozhen., 18, pp.176-85.(in Russian)

[22] Saka, B., (2015), A quartic B-spline collacation method for solving the nonlinear Schrodinger equation, Appl. Comput. Math., 14(1), pp.75-86.

[23] Shokri, A., (2015), An explicit trigonometrically fitted ten-step method with phase-lag of order infinity for the numerical solution of the radial Schrodinger equation, Appl. Comput. Math., 14(1), pp.63-74.

[24] Stashevskaya, V.V., (1953), On inverse problems of spectral analysis for a class of differantial equations, Dokl. Akad. Nauk. SSSR, 93, pp.409-411.

[25] Yurko, V. A., (1980), On stability of recovering the Sturm-Liouville operators, Diff. Eqns., Theory Funct. (Saratov Univ.), 3, pp.113-24.(in Russian).

[26] Watson, G.A., (1962), A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge.

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